FALL 2024: MATH 790 HOMEWORK

Homework problems from *Linear Algebra Done Right* will be labelled LADR.

HW 1. Let V be a vector space over the field F. Note that we are not assuming that V is finite dimensional.

(i) Prove the following version of the exchange property. Let $\{u_1, \dots, u_n\} \subseteq V$ and set $U := \langle u_1, \dots, u_n \rangle$. Suppose $v_1, \dots, v_m \in U$ are linearly independent. Prove that $m \leq n$.
(ii) Give a detailed prove using Zorn's lemma to show that any vector space has a basis.
(iii) Assume that $F = \mathbb{C}$. Prove that V is also a vector space over $\mathbb{R}$, and assuming V is finite dimensional over $\mathbb{C}$, find the dimension of V as a vector space over $\mathbb{R}$ in terms of the dimension of V over $\mathbb{C}$.
(iv) Let $W \subseteq V$ be a subspace. Use Zorn's lemma to prove there exists a subspace $U \subseteq V$ maximal with respect to the property that $W \cap U = 0$.

HW 2. Let V be a vector space over the field F.

(i) Suppose $T : V \to W$ is a linear transformation between finite dimensional vectors spaces. Assume α_1, α_2 are bases for V and β_1, β_2 are bases for W. Use the crucial formula from the lecture of August 28 to write a formula relating the matrices $[T]_{\alpha_1}^{\beta_1}$ and $[T]_{\alpha_2}^{\beta_2}$.
(ii) Prove that if the dimension of V equals n, with $n > 0$, then there cannot exist a chain of subspaces $(0) \subsetneq W_1 \cdots \subsetneq W_n \subsetneq V$. Conclude that if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is an ascending chain of subspaces of V, then there exists $n_0 \geq 1$ such that $U_s = U_{n_0}$, for all $s \geq n_0$.
(iii) Suppose F is infinite. Prove that V is not the union of finitely many proper subspaces of V.

HW 3. This homework uses the notation from the second day of class, as it appears in the Daily Update from August 28. Let A be an $n \times n$ matrix with coefficients in the field F.

(i) Let $T : V \to W$ be a linear transformation and set $A = [T]_\alpha^\beta$. For $v \in V$ let $[v]_\alpha$ denote the $n \times 1$ column vector in F^n obtained as follows: If $v = a_1 v_1 + \cdots + a_n v_n$, then $[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. The vector $[T(v)]_\beta$ in W is defined similarly. Prove that $[T(v)]_\beta = A \cdot [v]_\alpha$.
(ii) Show that if B is obtained from A by interchanging two rows, then $|B| = -|A|$.
(iii) Let E be an elementary matrix, i.e., an $n \times n$ matrix obtained from I_n by applying an elementary row operation. Prove that EA is obtained from A by apply the same elementary row operation to A.
(iii) Prove that if A is any matrix, then there is a sequence of elementary row operations that put A into *reduced row echelon form.*
(iv) Show that if E is an elementary matrix corresponding to an elementary row operation of a given type, then E^t is an elementary matrix corresponding to a row operation of the same type.

HW 4. For problems these you may use any of the properties of the determinant discussed in class.

(i) For an $n \times n$ matrix A, verify the Laplace expansion along the kth row: $|A| = \Sigma_{j=1}^n (-1)^{j+k} a_{kj} |A_{kj}|$.
(ii) Let A be an $n \times n$ invertible matrix such that every entry is ± 1. Prove that $|A|$ is an integer divisible by 2^{n-1}.
(iii) Suppose that A and B are $(2k+1) \times (2k+1)$ matrices over $\mathbb{R}$ such that $AB = -BA$. Prove that A and B cannot both be invertible.

HW 5. Let V be a vector space of dimension n over the field F.

(i) Prove that the vector spaces $\mathcal{L}(V, V)$ and $\mathrm{M}_n(F)$ are isomorphic.
(ii) Using the Cayley-Hamilton theorem for matrices, prove that $\chi_T(T) = 0$, for $T \in \mathcal{L}(V, V)$.
(iii) For $f(x) \in F[x]$, with s the degree of $f(x)$, prove that $|xI_s - C(f(x))| = f(x)$, where $C(f(x))$ is the companion matrix of $f(x)$. In other words $\chi_{C(f(x))}(x) = f(x)$.

HW 6. In the first two problems below, V is a vector space of dimension n and $B := \{v_1, \ldots, v_n\} \subseteq V$ is a basis for V.

1. For $v \in V$, write $v = a_1v_1 + \cdots + a_nv_n$. Define $[v]_B := \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, a column vector in F^n.

   (i) For $\alpha, \beta \in F$ and $v, w \in V$, show that $[\alpha v + \beta w]_B = \alpha[v]_B + \beta[w]_B$.

   (ii) For $T \in \mathcal{L}(V, V)$ and $v \in V$, show that $[T(v)]_B = [T]_B^B \cdot [v]_B$.

2. Suppose $\lambda \in F$ and $T \in \mathcal{L}(V, V)$. Using the corresponding result for matrices, prove that λ is an eigenvalue of T if and only if $\chi_T(\lambda) = 0$.

3. Prove a uniqueness statement for the division algorithm in $F[x]$, i.e., prove that if $f(x), g(x), h(x), r(x), h_0(x), r_0(x)$ are in $F[x]$ and

$$g(x) = f(x)h(x) + r(x) = f(x)h_0(x) + r_0(x),$$

where $r(x)$, $r_0(x)$ are either zero or have degree less than the degree of $f(x)$, then $h(x) = h_0(x)$ and $r(x) = r_0(x)$.

HW 7. 1. Let $W \subseteq \mathbb{R}$ be a plane through the origin and $L \subseteq \mathbb{R}^3$ a line through the origin, with $L \not\subseteq W$. Prove that $\mathbb{R}^3 = W \oplus L$.

2. Suppose $V = W_1 + W_2$, for proper non-zero subspaces $W_1, W_2 \subseteq V$. Prove there exists a subspace $U_2 \subseteq W_2$ such that $V = W_1 \oplus U_2$.

3. Suppose D is an $n \times n$ diagonal matrix with diagonal entires $\lambda_1, \ldots, \lambda_n$. Prove that $\lambda_1, \ldots, \lambda_n$ are the only eigenvalues of D.

HW 8. 1. Suppose $B' = \{v_1', \ldots, v_n'\}$ is a basis for V and $P = (p_{ij})$ an $n \times n$ matrix over F. Consider $B = \{v_1, \ldots, v_n\}$, where each $v_i = p_{i1}v_1' + \cdots + p_{in}v_n'$. Show that B is a basis for V if and only if P is an invertible matrix.

2. Transcribe the main theorem concerning diagonalizability of linear transformations presented at the end of the lecture on September 13 to a statement about diagonalizability for matrices, and then use the linear transformation form of the theorem to prove the matrix form of the theorem.

HW 9. 1. Suppose B is a basis for the vector space V and $B = B_1 \cup \cdots \cup B_r$ is a partition of B. Set $W_i := \text{Span}(B_i)$, for each $1 \leq i \leq r$. Show that $V = W_1 \oplus \cdots \oplus W_r$. Note: V need not be finite dimensional, though you can assume this initially, to get a feeling for how this works.

2. Let $T : F^n \to F^n$ be a linear transformation, suppose $E \subseteq F^n$ is the standard basis, and write $A = [T]_E^E$. Suppose P is an invertible matrix such that $P^{-1}AP = D$, where D is a diagonal matrix. Let $C_1, \ldots, C_n$ be the columns of P, and set $B := \{C_1, \ldots, C_n\}$. Prove that B is a basis for F^n and $[T]_B^B = D$.

3. Let F be a field and $T_A : F^2 \to F^2$ be the linear transformation whose matrix with respect to the standard basis is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Determine if T_A is diagonalizable over the fields: (a) $F = \mathbb{R}$, (b) $F = \mathbb{C}$, (c) $F = \mathbb{Z}_2$, and (d) $F = \mathbb{Z}_3$.

4. Let $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis is $B = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}$. Show that T_B is diagonalizable. Find an invertible 2×2 matrix P such that $P^{-1}BP$ has the eigenvalues of B down its diagonal.

Henceforth, whenever we say that V is an inner product space, we assume that $F = \mathbf{C}$ or $\mathbb{R}$.

HW 10. 1. Let V denote the vector space of complex polynomials having degree less than or equal to n. For $f, g \in V$, set $\langle f(x), g(x) \rangle := \int_{-1}^{1} f(x)\overline{g(x)}\, dx$. Show that this defines an inner product on V.

2. Suppose V is an inner product space Show:

   (i) $||v|| = 0$ if and only if $v = \vec{0}$.

   (ii) $||\lambda v|| = |\lambda| \cdot ||v||$, for all $v \in V$ and $\lambda \in F$. Here for the complex number λ, $|\lambda| := \sqrt{a^2 + b^2}$, if $\lambda = a + bi$. Note if $\lambda \in \mathbb{R}$, $|\lambda|$ is just the absolute value of λ.

(iii) If $\vec{0} \neq v \in V$, and $\lambda = \frac{1}{||v||}$, show that $||\lambda v|| = 1$.

3. Suppose V is an inner product space defined over $\mathbb{R}$. Show:
   (i) $\langle u+v, u-v\rangle = ||u||^2 - ||v||^2$, for all $u, v \in V$.
   (ii) If $u, v \in V$ have the same length, then $u+v$ is orthogonal to $u-v$.
   (iii) The two diagonals of any rhombus are perpendicular to each other.

HW 11. 1. Let V be the vector space of real polynomials of degree less than or equal to two. Define $\langle f(x), g(x)\rangle := \int_0^2 f(x)g(x)$. Find an orthonormal basis for V.

2. Let W be a subspace of the finite dimensional inner product space V. Show that $V = W \oplus W^{\perp}$. Hence the name *orthogonal complement* for $W^{\perp}$. Hint: Start with an orthogonal basis for W and extend it to an orthogonal basis for V. Thus, every vector $v \in V$ can we written uniquely as $v = w + w'$, with $w \in W$ and $w' \in W^{\perp}$. The vector w is called the *orthogonal projection* of v onto W.

HW 12. 1. Let $C = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$. Find an orthogonal matrix Q such that $Q^{-1}CQ$ is diagonal. Now, Let $T_C : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{R}^3$ is C. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for T.

2. Find a 2×2 matrix over $\mathbb{R}$ that is diagonalizable, but not orthogonally diagonalizable.

HW 13. Let V be a finite dimensional inner product space over $\mathbf{C}$.

1. For $T \in \mathcal{L}(V, V)$, using the definition of T^* from class, prove a version of the first spectral theorem over $\mathbf{C}$ for T, using the corresponding result for matrices.

2. For $T, S \in \mathcal{L}(V, V)$, use the definition of T^* from class to prove the following properties:
   (i) $(S+T)^* = S^* + T^*$.
   (ii) $(ST)^* = T^*S^*$.
   (iii) $\lambda T)^* = \overline{\lambda} T^*$.
   (iv) $(T^*)^* = T$.
   (v) If T is invertible, $(T^{-1})^* = (T^*)^{-1}$.

HW 14. Give an example of a matrix $A \in \mathrm{M}_2(\mathbf{C})$ that is not self-adjoint, but A is normal, and its entries are not in $\mathbb{R}$. Then show that, for you particular choice of A, $||Av|| = ||A^*v||$, for all $v \in \mathbf{C}^2$.

HW 15. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Show that these are normal matrices. Then show that when A acts on $\mathbb{R}^4$ by multiplication, there are four invariant subspaces, whereas when B acts on $\mathbb{R}^4$ by multiplication, there are infinitely many invariant subspaces. The difference here lies in the difference between $\mu_A(x)$ and $\mu_B(x)$. Calculate these polynomials.

HW 16. 1. Let A be an $m \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$. Prove that: (a) A^*A and AA^* have the same eigenvalues, counted with multiplicity and (b) A^*A and A have the same rank.

2. Find the singular value decomposition for the following matrices: (a) $A = \begin{pmatrix} i & 2i \\ 3i & 6i \end{pmatrix}$; (b) $B = \begin{pmatrix} 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

HW 17. Let $F[x]$ denote the ring of polynomials with coefficients in the field F.

(i) Let $p(x) \in F[x]$ be a non-constant irreducible polynomial. Prove that for any non-constant $f(x)$ in $F[x]$, the GCD of $p(x), f(x)$ is either $p(x)$ or 1.

(ii) Show that if $p(x)$ is irreducible over F and $p(x)$ divides $f(x) \cdot g(x)$, then $p(x)$ divides $f(x)$ or $p(x)$ divides $g(x)$. (Hint: Use (i) and Bezout's Principle.)

(iii) Prove that if $p_1(x) \cdots p_r(x) = q_1(x) \cdots q_s(x)$, and each $p_i(x), q_j(x)$ is monic and irreducible over F, then $r = s$, and after re-indexing, $q_i(x) = \cdot p_i(x)$. In other words, the factorization property for polynomials in $F[x]$ is in fact a *unique factorization* property.

HW 18. 1. Consider $f(x) = x^4 + x^3 + x + 1$ and $x^4 + 2x$ in $\mathbb{Z}_2[x]$. Use the Euclidean algorithm to find the GCD of $f(x)$ and $g(x)$, then write this GCD as $a(x)f(x) + b(x)g(x)$, for some $a(x), b(x) \in \mathbb{Z}_2[x]$.

2. Consider the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ as an element of $\mathrm{M}_2(\mathbb{R})$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$. Find a basis $B \subseteq \mathbb{R}^3$ such that the matrix of T with respect to B is block diagonal, with one block a 2×2 companion matrix and the other block a 1×1 matrix.

3. Suppose A is a 3×3 matrix over $\mathbb{R}$ whose minimal polynomial equals x^3. Show there is an invertible matrix P such that $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hint: What form must $\mu_{A,v}(x)$ take, for $v \in \mathbb{R}^3$.

HW 19. 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis is $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}$.

(i) For $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, show that $\langle T, v \rangle = \mathbb{R}^3$.

(ii) For $w = (T^2 + I)(v)$, find $\mu_{T,w}(x)$ and determine $\langle T, w \rangle$,

2. Suppose $T \in \mathcal{L}(V, V)$ and there are non non-trivial T-invariant subspaces of V. Prove V is a T-cyclic vector space.

3. Assume $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $\langle T, v \rangle = \mathbb{R}^3$, for some $v \in \mathbb{R}^3$. Let N be the number of T-cyclic subspaces. Show that $N = 2, 4,$ or 6.

HW 20. 1. Let $A \in \mathrm{M}_3(\mathbb{R})$. Show that $\mu_A(x)$ cannot be an irreducible polynomial of degree two.

2. Let $E = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$ be the standard basis and suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $[T]_E^E = \begin{pmatrix} -1 & 3 & -2 \\ -1 & 3 & -4 \\ -1 & 1 & -2 \end{pmatrix}$.

(i) Find $\mu_{T,e_i}(x)$, for each e_1, e_2, e_3.

(ii) Compute $\mu_T(x)$.

(iii) Find a maximal vector for $\mathbb{R}^3$ with respect to T.

3. Do the same as in 3, for the matrix $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

HW 21. 1. Use the invariant factor form of the RCF theorem to prove the elementary divisor form of the RCF theorem, as stated in the lecture of October 21.

2. Prove a matrix version of the elementary divisor form of the RCF theorem.

HW 22. For the two matrices given in HW 20, find their rational canonical forms and the corresponding change of basis matrices.

HW 23. 1. For $p \geq 1$, find p distinct pth roots of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2. Find the solution to the system of first order linear differential equations given by the vector equation $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, with initial condition $\mathbf{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Here $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

HW 24. Prove the third isomorphism theorem: Let U, W be subspaces of the vector space V. Prove that $(U + W)/W$ is isomorphic to $U/(U \cap W)$. Hint: Find a well-defined surjective linear transformation from $U \to (U + W)/W$ and then apply the First Isomorphism Theorem.

2. Let V and U be vector spaces and $W \subseteq V$ a subspace. Set $K := \{f \in \mathcal{L}(V, U) \mid W \subseteq \text{kerne}(f)\}$. Show that K is a subspace of $\mathcal{L}(V, U)$ and $\mathcal{L}(V, U)/K \cong \mathcal{L}(V/W, U)$.

HW 25. 1. Given vector spaces V, W, suppose (P, f) is a tensor product of V and W. Suppose $\alpha : P \to P_1$ is an isomorphism of vector spaces. Set $f_1 := \alpha \circ f$. Show that (P_1, f_1) is a tensor product of V and W.

2. Let L, M be vector spaces over F. Suppose that $T : L \to M$ and $S : M \to L$ are linear transformations sch that ST is the identity on L and TS is the identity on M. Prove that T is an isomorphism with inverse S.

3. For vector spaces V, W_1, W_2 over F prove that $V \otimes (W_1 \oplus W_2) \cong (V \otimes W_1) \oplus (V \otimes W_2)$.